Role of Linear Algebra and its Applications in Health Care Monitoring

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Abstract: Many problems of applications require solving of large system of equations, either under-determined, or determined, or over-determined. The equations may be subjected to constraints. The systems are typically large and sparse systems, wherein the entries of the matrix are predominantly zero. In this review article, we stress upon the applications of solving large systems arising in transmission and emission tomography. Because the measured data is typically insufficient to give a unique solution, optimization techniques such as least-squares method can be used. If the number of equations and the number of variables are small then we can solve the system using Gauss elimination method. It is quite natural in problems of applications, such as medical imaging, to encounter large system of linear equations. Thus, it is common to prefer inexact solutions over exact ones. Even when the number of equations and unknowns is large, there may not be enough information to obtain a unique solution. This is the case of over-determined system of equations and is quite normal in medical tomographic imaging, in which the images are artificially discretized approximations of parts of the interior of the body.

Keywords: Radiography, Tomography, Inconsistent system of Linear Equations, Method of Least Squares

I. A BRIEF HISTORY

In 1895, Roentgen discovered X-rays and pioneered medical imaging. His initial publication contained a radiograph (i.e. an X-ray generated photograph) of Mrs. Roentgen’s hand. For the first time, it was possible to visualize non-invasively (i.e., not through surgery) the interior of the human body. The discovery was widely publicized in the press and an “X-ray mania” prevailed in Europe and the United States. Within only a few months, public demonstrations were organized, commercial ventures were created and innumerable medical applications were investigated. The field of radiography took its birth. See [16].

II. ROLE OF LINEAR ALGEBRA

A fundamental question: what is linear algebra? As it is clear, it comprises of two words, “linear” and “algebra”, which means addition and multiplication of “straight” and/or “flat” objects, in higher dimensions, without actual visualization. This algebraic approach keeps geometry in the background and the main purpose is to solve either deterministic or undeterministic systems of linear equations.

Linear Algebra is a blend of both analytical geometry and algebra. It is an evolution from the lower division mathematics to the upper division. Although abstract in nature, the approach is axiomatic. The subject helps in developing geometric instincts. The “geometric intuition” is then translated into an “algebraic picture” and vice versa. It brings further clarity in thoughts due to abstract notions and stimulates the thinking power. It also links many branches of mathematics in a natural way. For basic ideas refer to [1] – [5].

III. INTRODUCTION

Computer-assisted tomography (CAT) scans have brought in a revolution in medical practice. Refer to Figure 1. One example of CAT is transmission tomography (TT). The goal is to image the spatial distribution of radiation attenuation. The term “tomography” itself is derived from the Greek word “tomos”, meaning part or slice. Tomography refers to imaging by sections or sectioning, through the use of any kind of penetrating wave. A device used in tomography is called a toigraph, while the image produced is a tomogram. Tomography as the computed tomographic (CT) scanner was invented by Sir Godfrey Hounsfield, and thereby made an exceptional contribution to medicine. The method is used in radiology, archaeology, biology, geophysics, oceanography, materials science, astrophysics, quantum Information, and other sciences. In most cases it is based on the mathematical procedure called tomographic reconstruction. Refer to [6] – [8].

Transmission Tomography is commonly associated with medical diagnosis. It also finds applications in determining the sound speed profile in the ocean. Previously “CAT scan” was referred to just x-ray transmission tomography. But now the term is used to describe any of the scanning modalities in medicine, including single-photon emission computed tomography (SPECT), positron emission tomography (PET), ultrasound, and magnetic resonance imaging (MRI).
There are two modes of scanning. The first one is the parallel mode and the other is the fan-beam mode. Refer to figure 3. Here the intensities of the X-ray beams are measured, after they pass through the cross section, by an X-ray detector, and these measurements are relayed to a computer where they are processed. The early CT devices are effectively parallel-beam scanners, but the advent of fan-beam detectors demands more sophisticated mathematics to describe the reconstruction process.

IV. THE MATHEMATICAL BASIS

The mathematical basis for tomographic imaging was laid down by Johann Radon. It is applied in Computed Tomography to obtain cross-sectional images of patients. [9] – [11] and [20].

Figure 1: CT System for Head scanning

The projection of an object at a given angle $\theta$ is made up of a set of line integrals. In X-ray CT, the line integral represents the total attenuation of the beam of x-rays as it travels in a straight line through the object. The resulting image is a 2D model of the attenuation coefficient. A simplest and easiest way to visualise this method of scanning is imaging through slicing (refer Figure 2). Here we consider the data to be collected as a series of parallel rays, at various positions $\gamma$, across a projection, at an angle $\theta$ and this procedure is repeated for various angles. It is known that the attenuation occurs exponentially in tissues. Refer [5], [13] – [15], [17] – [19].

V. TRANSMISSION TOMOGRAPHY

In TT, radiation, normally x-ray, is transmitted through the object being scanned. The object of interest need not be a living human being. The x-ray beams are assumed to travel along straight lines through the object. The initial intensity of the beams is known and the intensity of the exiting beams is measured along each line. The goal is to estimate the image of the x-ray attenuation function, which closely correlates with the spatial distribution of the attenuating material within the object. Unexpected absence of attenuation can indicate an anomaly in the anatomy.

Due to the presence of various attenuating materials within the body, the x-ray beam weakens as it travels along its line through the body. The reduced intensity of the exiting beam provides a measure of how much attenuation the x-ray encountered as it traveled along the line. But we will have lost the information of where along the line it encountered the attenuation. In other words, we know the integral of the attenuation function along that line. It is just by repeating the process with other beams along other lines that one can localize the attenuation and then reconstruct the image of this non-negative attenuation function. In some of the approaches, the lines are all in the same plane and thus a reconstruction of a single slice through the object is the goal; in other cases, a fully three dimensional scanning occurs. Refer to figure 4. [12], [16].

In theory, we consider the integral of the attenuation function along every line through the object. The Radon transform assigns to each attenuation function, its integral over every line. The mathematical problem is then to obtain the inverse of the Radon transform, i.e. to recapture the attenuation function from a set of line integrals. One way is to use the Fourier transform proving the well-known Central Slice Theorem. The reconstruction is then inversion of the Fourier transform. There are various methods of inversion which rely on frequency-domain filtering and back-projection, which are beyond the scope of this article.

Theoretically, the data are line integrals of the function of interest. But in practice, we never have all the line integrals. Ultimately, we will construct a discrete image made up of finitely many pixels. Consequently, it can be assumed without loss in generality that the attenuation function to be estimated is well approximated by a function that is constant across small squares (or cubes), called pixels (or voxels), and that the goal is to determine these finitely many pixel values.

Each image consists of a fairly large number of little squares called pixels (picture elements). The matrix corresponding to a digital image assigns a whole number to each pixel. For example, in the case of a 256 x 256 pixel gray scale image, the image is stored as a 256 x 256 matrix, with each
element of the matrix being a whole number ranging from 0 (for black) to 255 (for white). The JPEG compression technique divides an image into 8 x 8 blocks and assigns a matrix to each block.

Images are represented in terms of grids of pixel values, that is, they become matrices, and then are represented as column vectors. Image processing is then a manipulation of these column vectors by matrix operations. This digitization means that very large systems of linear equations have to be solved. The need for fast algorithms to solve these large systems of linear equations turns linear algebra into a branch of applied and computational mathematics.

When the problem is thus discretized, different mathematics begins to play a role. The line integrals are replaced by finite sums, and the problem can be viewed as one of solving a large number of linear equations, subject to side constraints, such as the non-negativity of the pixel values. The Fourier transform and the Central Slice Theorem are still relevant, but in discrete form, with the Fast Fourier Transform (FFT) playing a major role in discrete filtered back-projection methods. This approach provides faster reconstruction, but is still limited. We can look for iterative algorithms for solving large systems of linear equations, subject to constraints. This approach allows for greater inclusion of the physics into the reconstruction.

The reconstruction of a density function from projections along lines reduces to the solution of the Radon transform. This was studied first in 1917 and it is today a basic tool in medical diagnosis, in tokamak (A tokamak is a device using a magnetic field to confine a plasma in the shape of a torus) monitoring, in plasma physics or for astrophysical applications. The reconstruction is also called tomography. Various mathematical tools developed for the solution of this problem leads to the construction of sophisticated scanners.

VI. DERIVING THE SYSTEM OF EQUATIONS

In this section, we shall see how the construction of a cross-sectional view of a human body, by analyzing X-ray scans, leads to an inconsistent linear system of equations. Refer [4] and [5].
By the beam density of the $i^{th}$ beam of a scan, denoted by $b_i$, we mean

$$b_i = \ln \left( \frac{\text{no. of photons entering the first pixel}}{\text{no. of photons leaving the last pixel}} \right)$$

$$= -\ln \left( \frac{\text{fraction of photons passing through the row of $n$ pixels without being absorbed}}{\text{no. of photons of the $i^{th}$ beam entering the detector without the cross section in the field of view}} \right)$$

$$= \ln \left( \frac{\text{fraction of photons of the $i^{th}$ beam that pass through the cross section without being absorbed}}{\text{no. of photons of the $i^{th}$ beam entering the detector with the cross section in the field of view}} \right)$$ (2)

Thus, if the $i^{th}$ beam passes squarely through a row of $n$ pixels, then it follows from Equations 1 and 2 that

$$x_1 + x_2 + \cdots + x_n = b_i$$

In this equation, $b_i$ is known from the clinical and calibration measurements, and $x_1$, $x_2$, ..., $x_n$, are unknown pixel densities that must be determined.

More generally, if the $i^{th}$ beam passes squarely through a row (or column) of pixels with numbers $j_1, j_2, \ldots, j_l$ then we have

$$x_{j_1} + x_{j_2} + \cdots + x_{j_l} = b_i$$

If we set

$$a_{ij} = \begin{cases} 1, & \text{if } j = j_1, j_2, \ldots, j_l \\ 0, & \text{otherwise} \end{cases}$$

then we may write this equation as

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$ (3)

This is referred to as the $i^{th}$ beam equation.

Referring to figure 4, however, we see that the beams of a scan do not necessarily pass through a row or column of pixels squarely. Instead, a typical beam passes diagonally through each pixel in its path. There are many ways to take this into account. Figure 5 explains three methods of defining the quantities $a_{ij}$ that appear in equation 3, each of which reduces to the previous definition when the beam passes squarely through a row or column of pixels. Each method is more exact than its predecessor, but computationally more difficult.

Using any one of the three methods to define the $a_{ij}$’s in the $i^{th}$ beam equation, we can write the set of $M$ beam equations in a complete scan as:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = b_2$$

$$\vdots$$

$$a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N = b_M$$

In this way we have a linear system of $M$ equations (the $M$ beam equations) in $N$ unknowns (the $N$ pixel densities). Depending on the number of beams and pixels used, we may have $M > N$, $M = N$, or $M < N$. Only the case when $M > N$ is considered, which gives rise to an over-determined system of equations, in which there are more beams in the scan than pixels in the field of view. Because of inherent modeling and experimental errors in the problem, we cannot expect the above linear system to have an exact mathematical solution for the pixel densities. The above over determined system can be solved using the well-known method of least squares.
VII. CONCLUSIONS

The occurrence of inconsistent system of linear equations, while tomographs are constructed from thousands of individual, hairline-thin X-ray beams that lie in the plane of the cross section, is reviewed. There are two modes of scanning. The first one is the parallel mode and the second is the fan-beam mode. In the parallel mode a single X-ray source and X-ray detector pair are translated across the field of view containing the cross section, and many measurements of the parallel beams are recorded. In the fan-beam mode of scanning, a single X-ray tube generates a fan of collimated beams whose intensities are measured simultaneously by an array of detectors on the other side of the field of view. The X-ray tube and the detector array are rotated through many angles, and a set of measurements is taken at each angle until the scan is completed. Thus Linear Algebra plays an important role in Health Care modeling.

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REFERENCES